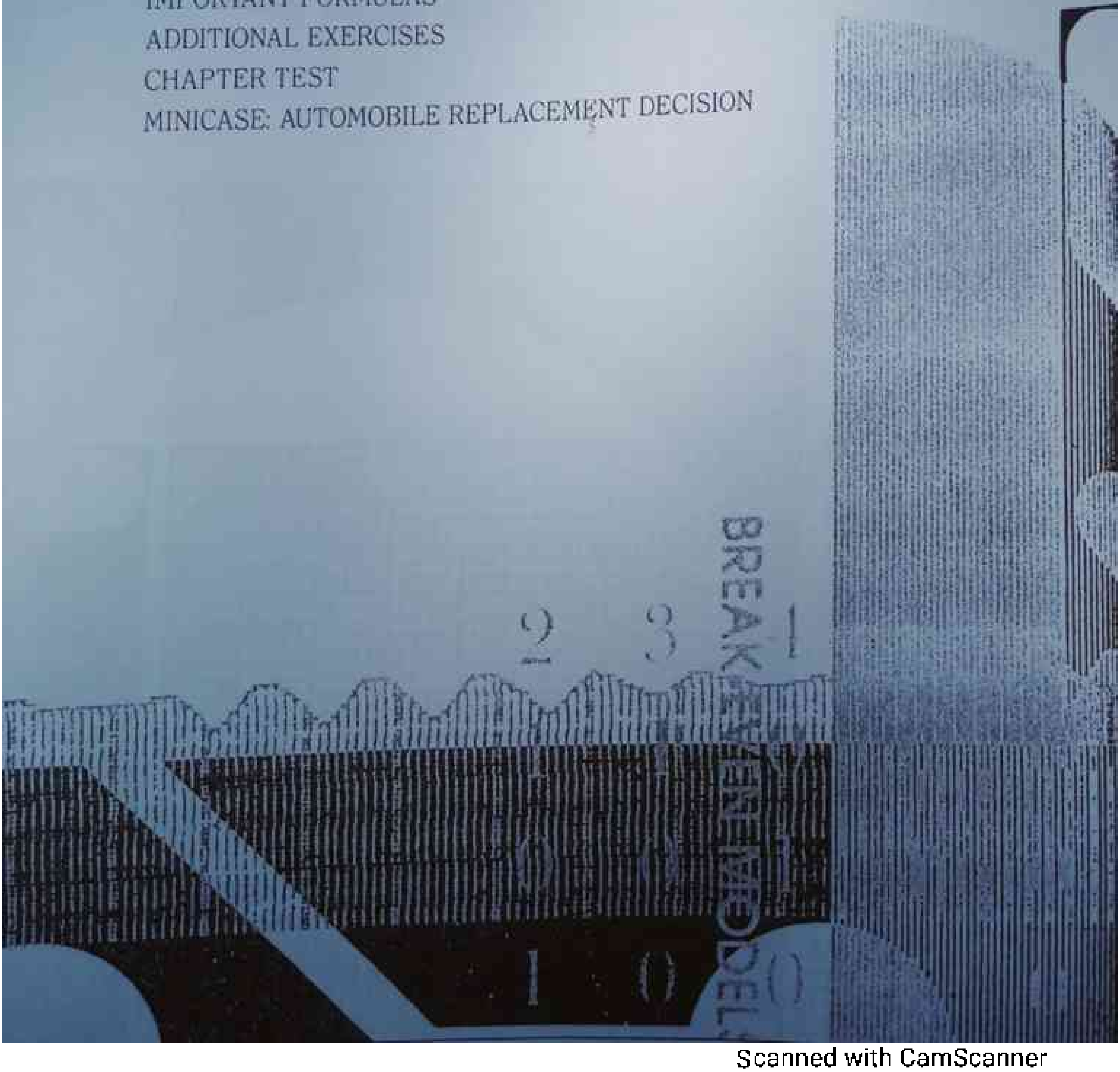
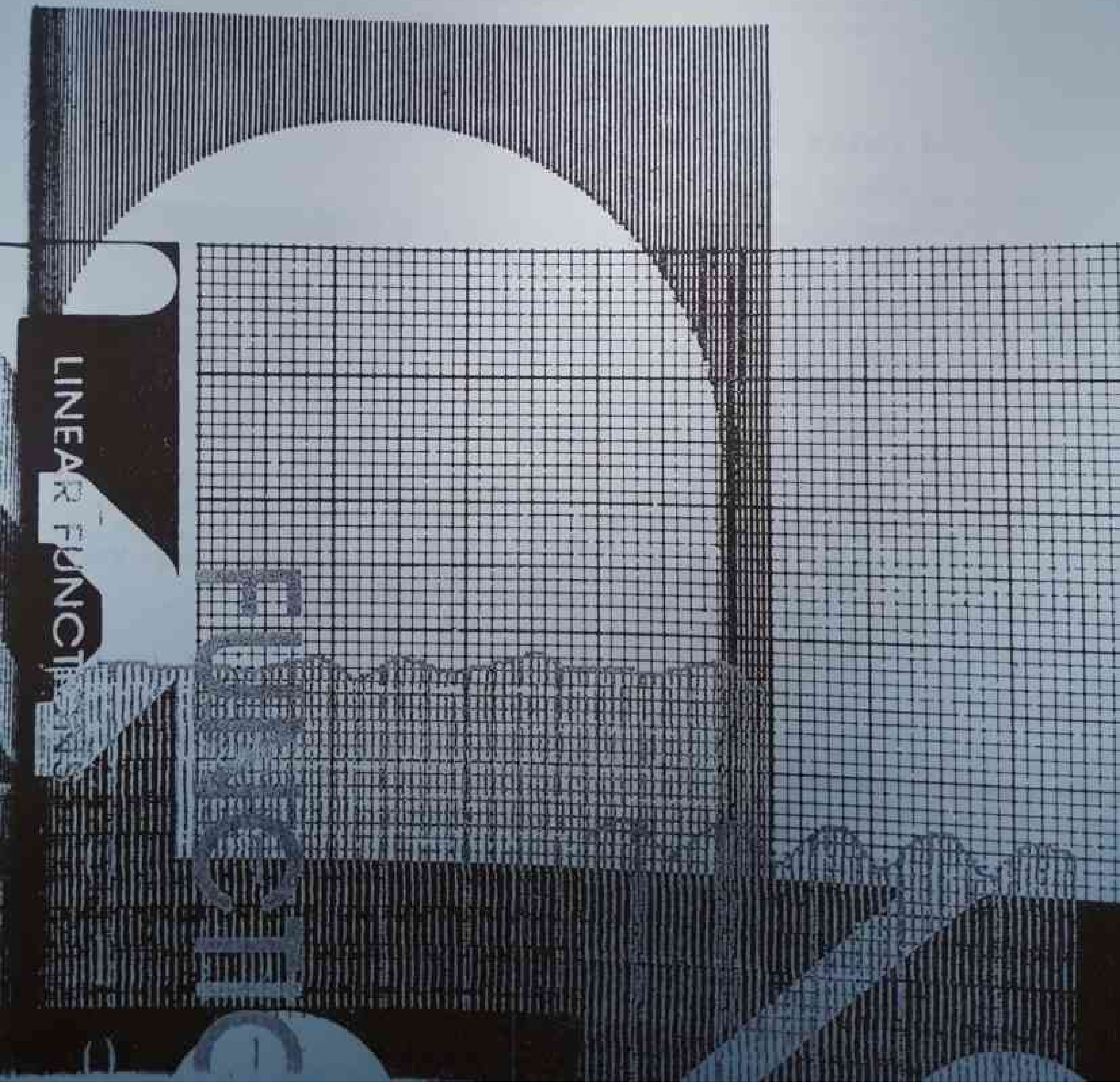
CHAPTER 5

LINEAR FUNCTIONS: APPLICATIONS



- Present a discussion of the characteristics of linear functions
- Present a wide variety of applications of linear functions



MOTIVATING SCENARIO: Federal Income Taxes

In 1990, the federal tax rates for a married couple filing jointly were as shown in the table.

Taxable Income		
Over	But not Over	Tax Rate
$ 0	$ 32,450	15%
32,450	78,400	28
78,400	162,770	33
162,770		28

What is desired is a formula, or set of formulas, which would enable a married couple to calculate their federal taxes once they know their taxable income. [Example 10]

In this chapter we extend the material presented in Chaps. 2 and 4 by presenting a discussion of linear functions. After reviewing the form and assumptions underlying these functions, we will see examples which illustrate the applications of these models in business, economics, and other areas.

5.1 LINEAR FUNCTIONS

General Form and Assumptions

DEFINITION: LINEAR FUNCTION INVOLVING ONE INDEPENDENT VARIABLE

A *linear function* f involving one independent variable x and a dependent variable y has the general form

$$y = f(x) = a_1x + a_0 \quad (5.1)$$

where a_1 and a_0 are constants, $a_1 \neq 0$.

Equation (5.1) should be familiar from the previous chapter. In addition, you should recognize this as the slope-intercept form of a linear equation with slope a_1 and y intercept occurring at $(0, a_0)$. *For a linear function having the form of Eq. (5.1), a change in the value of y is directly proportional to a change in the value of x.* This rate of change is constant and represented by the slope a_1.

Example 1 in Chap. 4 presented the linear salary function

$$y = f(x) = 3x + 25$$

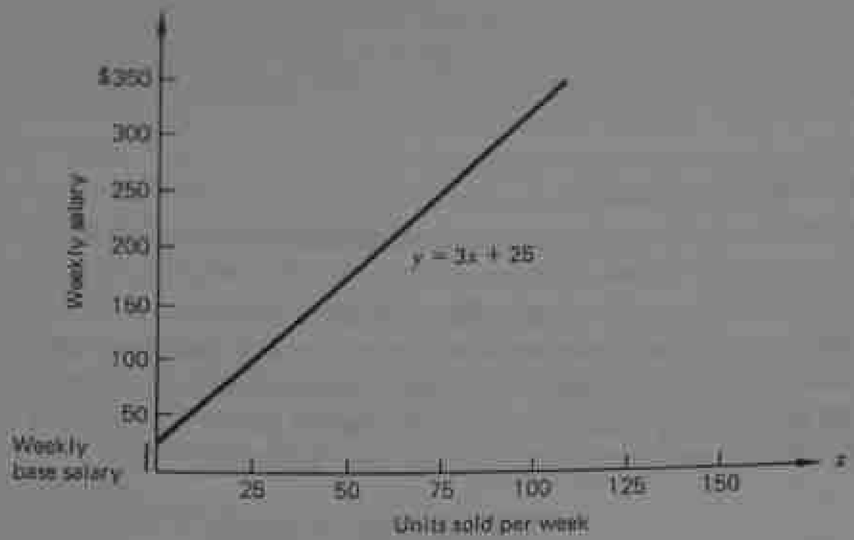


Figure 5.1 Linear salary function.

where y is defined as weekly salary in dollars and x represents the number of units sold per week. In this salary function, the salesperson is paid a base salary of \$25 per week and a commission of \$3 per unit sold. The change in the person's weekly salary is directly proportional to the change in the number of units sold. That is, the slope of 3 indicates the increase in weekly salary associated with each additional unit sold. The graph of the salary function appears in Fig. 5.1. Note that the graph is in the first quadrant, restricting x and y to nonnegative values. Does this make sense?

DEFINITION: LINEAR FUNCTION INVOLVING TWO INDEPENDENT VARIABLES

A linear function f involving two independent variables x_1 and x_2 and a dependent variable y has the general form

$$y = f(x_1, x_2) = a_1x_1 + a_2x_2 + a_0 \quad (5.2)$$

where a_1 and a_2 are (nonzero) constants and a_0 is a constant.

For a linear function of the form of Eq. (5.2) the variable y depends jointly on the values of x_1 and x_2. The value of y varies in direct proportion to changes in the values of x_1 and x_2. Specifically, if x_1 increases by 1 unit, y will change by a_1 units. And if x_2 increases by 1 unit, y will change by a_2 units.

EXAMPLE 1 Assume that a salesperson's salary depends on the number of units sold of each of two products. More specifically, assume that the salary function

$$y = f(x_1, x_2)$$

is

$$y = 5x_1 + 3x_2 + 25$$

where y = *weekly salary*, x_1 = *number of units sold of product 1*, and x_2 = *number of units sold of product 2*. This salary function suggests a base weekly salary of \$25 and commissions per unit sold of \$5 and \$3, respectively, for products 1 and 2.

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DEFINITION: LINEAR FUNCTION INVOLVING n INDEPENDENT VARIABLES

A linear function f involving n independent variables $x_1, x_2, \ldots, x_n$ and a dependent variable y has the general form

$$y = f(x_1, x_2, \ldots, x_n)$$

or

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_0 \qquad (5.3)$$

where $a_1, a_2, \ldots, a_n$ are (nonzero) constants and a_0 is a constant.

Linear Cost Functions

Organizations are concerned with *costs* because they reflect dollars flowing out of the organization. These outflows usually pay for salaries, raw materials, supplies, rent, heat, utilities, and so forth. As mentioned earlier, accountants and economists often define total cost in terms of two components: ***total variable cost*** and ***total fixed cost***. These two components must be added to determine total cost. The cost function for owning and operating the patrol car in Example 3 of Chap. 4 is an example of a linear cost function. The cost function,

$$C(x) = 0.40x + 18{,}000$$

had variable costs which varied with the number of miles driven and fixed costs of \$18,000.

Total variable costs vary with the level of output and are computed as the product of **variable cost per unit of output** and the level of output. In a production setting, variable cost per unit is usually composed of raw material and labor costs. In the example of the patrol car, variable cost per mile consisted of operating costs per mile such as gasoline, oil, maintenance costs, and depreciation.

Linear cost functions are very often realistic, although they ignore the possibility of ***economies*** or ***diseconomies of scale***. That is, linear cost functions imply ***constant returns to scale***. Constant returns to scale imply that regardless of the number of units produced, the variable cost for each unit is the same. This assumption ignores the possibility that the elements of the production process (laborers or machines) may become more efficient as the number of units produced increases or that buying raw materials in large quantities may result in quantity discounts which in turn may lower the variable cost per unit produced (this is an example of economies of scale). The cost function for the patrol car assumes that operating costs per mile will be \$0.40 regardless of the number of miles driven. We might expect that over the life of a piece of equipment, such as the patrol car, it will become less efficient and will require greater maintenance. This should translate into a higher variable cost per unit. Some cost models recognize these potential "nonlinearities" by using some measure of *average variable cost per unit*. In other situations a set of linear cost functions might be developed, each appropriate in certain cases depending on the level of output selected.

The following example illustrates the formulation of a linear cost function.

EXAMPLE 2 A firm which produces a single product is interested in determining the function that expresses annual total cost y as a function of the number of units produced x. Accountants indicate that fixed expenditures each year are \$50,000. They also have estimated that raw material costs for each unit produced are \$5.50, and labor costs per unit are \$1.50 in the assembly department, \$0.75 in the finishing room, and \$1.25 in the packaging and shipping department.

The total cost function will have the form

$$\begin{aligned} y &= C(x) \\ &= \text{total variable cost} + \text{total fixed cost} \end{aligned}$$

Total variable costs consist of two components: raw material costs and labor costs. Labor costs are determined by summing the respective labor costs for the three departments. Total cost is defined by the function

$$y = \text{total raw material cost} + \text{total labor cost} + \text{total fixed cost}$$

$$= \begin{matrix}\text{total raw}\\ \text{material cost}\end{matrix} + \begin{matrix}\text{labor cost}\\ \text{(assembly dept.)}\end{matrix} + \begin{matrix}\text{labor cost}\\ \text{(finishing room)}\end{matrix} + \begin{matrix}\text{labor cost}\\ \text{(shipping dept.)}\end{matrix} + \begin{matrix}\text{total fixed}\\ \text{cost}\end{matrix}$$

or $$y = 5.50x + (1.50x + 0.75x + 1.25x) + 50{,}000$$

which simplifies to

$$y = f(x) = 9x + 50{,}000$$

The 9 represents the combined variable cost per unit of \$9.00. That is, for each additional unit produced, total cost will increase by \$9.

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Linear Revenue Functions

The money which flows into an organization from either selling products or providing services is often referred to as ***revenue***. The most fundamental way of computing total revenue from selling a product (or service) is

$$\text{Total revenue} = (\text{price})(\text{quantity sold})$$

An assumption in this relationship is that the selling price is the same for all units sold.

Suppose a firm sells n products. If x_j equals the number of units sold of product j and p_j equals the price of product j, the function which allows you to compute total revenue from the sale of the n products is

$$R = p_1x_1 + p_2x_2 + p_3x_3 + \cdots + p_nx_n \tag{5.4}$$

This revenue function can be stated more concisely using ***summation notation*** as

$$R = \sum_{j=1}^{n} p_jx_j \tag{5.5}$$

Those of you seeing summation notation for the first time may want to refer to Appendix B for an introduction to this concept.

EXAMPLE 3 A local car rental agency, Hurts Renta-Lemon, is trying to compete with some of the larger national firms. Management realizes that many travelers are not concerned about frills such as windows, hubcaps, radios, and heaters. I. T. Hurts, owner and president of Hurts, has been recycling used cars to become part of the fleet. Hurts has also simplified the rental rate structure by charging a flat \$9.95 per day for the use of a car. Total revenue for the year is a linear function of the number of car-days rented out by the agency, or *if* R = *annual revenue in dollars* and d = *number of car-days rented during the year*,

$$R = f(d) = 9.95d$$

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Linear Profit Functions

Profit for an organization is the difference between total revenue and total cost. Stated in equation form,

$$\text{Profit} = \text{total revenue} - \text{total cost} \tag{5.6}$$

If $\qquad$ Total revenue $= R(x)$

and $\qquad$ Total cost $= C(x)$

where x equals the quantity produced and sold, then profit is defined as

$$P(x) = R(x) - C(x) \tag{5.7}$$

When total revenue exceeds total cost, profit is positive. In such cases the profit may be referred to as a **net gain,** or **net profit.** When total cost exceeds total revenue, profit is negative. In such cases, the negative profit may be referred to as a **net loss,** or **deficit.** When the revenue and cost are linear functions of the same variable(s), the profit function is a linear function of the same variable(s).

EXAMPLE 4 A firm sells a single product for \$65 per unit. Variable costs per unit are \$20 for materials and \$27.50 for labor. Annual fixed costs are \$100,000. Construct the profit function stated in terms of x, the number of units produced and sold. What profit is earned if annual sales are 20,000 units?

SOLUTION

If the product sells for \$65 per unit, total revenue is computed by using the linear function

$$R(x) = 65x$$

Similarly, total annual cost is made up of material costs, labor costs, and fixed costs:

$$C(x) = 20x + 27.50x + 100{,}000$$

which reduces to the linear cost function

$$C(x) = 47.50x + 100{,}000$$

Thus the profit function is computed as

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 65x - (47.50x + 100{,}000) \\ &= 17.50x - 100{,}000 \end{aligned}$$

Notice that $P(x)$ is a linear function. The slope of 17.50 indicates that for each additional unit produced and sold, total profit increases by \$17.50. In business and economics, this is referred to as the **marginal profit** (the addition to total profit from selling the next unit).

If the firm sells 20,000 units during the year,

$$\begin{aligned} P(20{,}000) &= 17.50(20{,}000) - 100{,}000 \\ &= 350{,}000 - 100{,}000 \\ &= 250{,}000 \end{aligned}$$

EXAMPLE 5 **(Agricultural Planning)** A corporate agricultural organization has three separate farms which are to be used during the coming year. Each farm has unique characteristics which make it most suitable for raising one crop only. Table 5.1 indicates the crop selected for each farm, the annual cost of planting 1 acre of the crop, the expected revenue to be derived from each acre, and the fixed costs associated with operating each farm. In addition to the fixed costs associated with operating each farm, there are annual fixed costs of \$75,000 for the corporation as a whole. Determine the profit function for the three-farm operation if *x_j = the number of acres planted at farm j, r_j = revenue per acre at farm j, c_j = cost per acre at farm j, and F_j = fixed cost at farm j.*

TABLE 5.1

Farm	Crop	Cost/Acre (c_j)	Revenue/Acre (r_j)	Fixed Cost (F_j)
1	Soybeans	\$ 900	\$1,300	\$150,000
2	Corn	1,100	1,650	175,000
3	Potatoes	750	1,200	125,000

SOLUTION

Total revenue comes from the sale of crops planted at each of the three farms, or

$$\begin{aligned} R(x_1, x_2, x_3) &= r_1x_1 + r_2x_2 + r_3x_3 \\ &= 1{,}300x_1 + 1{,}650x_2 + 1{,}200x_3 \end{aligned}$$

Total costs are the sum of those at the three farms plus the corporate fixed costs, or

$$\begin{aligned} C(x_1, x_2, x_3) &= c_1x_1 + F_1 + c_2x_2 + F_2 + c_3x_3 + F_3 + 75{,}000 \\ &= 900x_1 + 150{,}000 + 1{,}100x_2 + 175{,}000 + 750x_3 + 125{,}000 + 75{,}000 \\ &= 900x_1 + 1{,}100x_2 + 750x_3 + 525{,}000 \end{aligned}$$

Total profit is a linear function computed as

$$\begin{aligned} P(x_1, x_2, x_3) &= R(x_1, x_2, x_3) - C(x_1, x_2, x_3) \\ &= 1{,}300x_1 + 1{,}650x_2 + 1{,}200x_3 - (900x_1 + 1{,}100x_2 + 750x_3 + 525{,}000) \\ &= 400x_1 + 550x_2 + 450x_3 - 525{,}000 \end{aligned}$$

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Section 5.1 Follow-up Exercises

1 Write the general form of a linear function involving five independent variables.

2 Assume that the salesperson in Example 1 (page 177) has a salary goal of $800 per week. If product B is not available one week, how many units of product A must be sold to meet the salary goal? If product A is unavailable, how many units must be sold of product B?

3 Assume in Example 1 (page 177) that the salesperson receives a bonus when combined sales from the two products exceed 80 units. The bonus is $2.50 per unit for each unit over 80. With this incentive program, the salary function must be described by two different linear functions. What are they, and when are they valid?

4 For Example 4 (page 181), how many units must be produced and sold in order to (*a*) earn a profit of $1.5 million, and (*b*) earn zero profit (break even)?

5 A manufacturer of microcomputers produces three different models. The following table summarizes wholesale prices, material cost per unit, and labor cost per unit. Annual fixed costs are $25 million.

	Microcomputer		
	Model 1	Model 2	Model 3
Wholesale price/unit	$500	$1,000	$1,500
Material cost/unit	175	400	750
Labor cost/unit	100	150	225

(*a*) Determine a joint total revenue function for sales of the three different microcomputer models.
(*b*) Determine an annual total cost function for manufacturing the three models.
(*c*) Determine the profit function for sales of the three models.
(*d*) What is annual profit if the firm sells 20,000, 40,000 and 10,000 units, respectively, of the three models?

6 For Example 5 (page 181), the board of directors has voted on the following planting program for the coming year: 1,000 acres will be planted at farm 1, 1,600 at farm 2, and 1,550 at farm 3.

(a) What are the expected profits for the program?

(b) A summer drought has resulted in the revenue yields per acre being reduced by 20, 30, and 10 percent, respectively, at the three farms. What is the profit expected from the previously mentioned planting program?

7 **Automobile Leasing** A car-leasing agency purchases new cars each year for use in the agency. The cars cost $15,000 new. They are used for 3 years, after which they are sold for $4,500. The owner of the agency estimates that the variable costs of operating the cars, exclusive of gasoline, are $0.18 per mile. Cars are leased for a flat fee of $0.33 per mile (gasoline not included).

(a) Formulate the total revenue function associated with renting one of the cars a total of x miles over a 3-year period.

(b) Formulate the total cost function associated with renting a car for a total of x miles over 3 years.

(c) Formulate the profit function.

(d) What is profit if a car is leased for 60,000 miles over a 3-year period?

(e) What mileage is required in order to earn zero profit for 3 years?

8 A company produces a product which it sells for $55 per unit. Each unit costs the firm $23 in variable expenses, and fixed costs on an annual basis are $400,000. If x equals the number of units produced and sold during the year:

(a) Formulate the linear total cost function.

(b) Formulate the linear total revenue function.

(c) Formulate the linear profit function.

(d) What does annual profit equal if 10,000 units are produced and sold during the year?

(e) What level of output is required in order to earn zero profit?

9 A gas station sells unleaded regular gasoline and unleaded premium. The price per gallon charged by the station is $1.299 for unleaded regular and $1.379 for unleaded premium. The cost per gallon from the supplier is $1.219 for unleaded regular and $1.289 for premium. If x_1 equals the number of gallons sold of regular and x_2 the number of gallons sold of premium:

(a) Formulate the revenue function from selling x_1 and x_2 gallons, respectively, of the two grades of gasoline.

(b) Formulate the total cost function from purchasing x_1 and x_2 gallons, respectively, of the two grades.

(c) Formulate the total profit function.

(d) What is total profit expected to equal if the station sells 100,000 gallons of unleaded regular and 40,000 of unleaded premium?

5.2 OTHER EXAMPLES OF LINEAR FUNCTIONS

In this section we will see, by example, other applications of linear functions.

EXAMPLE 6 **(Straight-Line Depreciation)** When organizations purchase equipment, vehicles, buildings, and other types of "capital assets," accountants usually allocate the cost of the item over the period the item is used. For a truck costing $20,000 and having a useful life of 5 years, accountants might allocate $4,000 a year as a cost of owning the truck. The cost allocated to any given period is called *depreciation*.

Accountants also keep records of each major asset and its current, or "book," value. For instance, the value of the truck may appear on accounting statements as \$20,000 at the time of purchase, \$20,000 − \$4,000 = \$16,000 1 year from the date of purchase, and so forth. Depreciation can also be thought of as the amount by which the book value of an asset has decreased.

Although there are a variety of depreciation methods, one of the simplest is ***straight-line depreciation.*** Under this method the rate of depreciation is constant. This implies that the book value declines as a linear function over time. *If V equals the book value (in dollars) of an asset and t equals time (in years) measured from the purchase date* for the previously mentioned truck,

$$V = f(t)$$
$$= \text{purchase cost} - \text{depreciation}$$

or

$$= 20{,}000 - 4{,}000t$$

The graph of this function appears in Fig. 5.2.

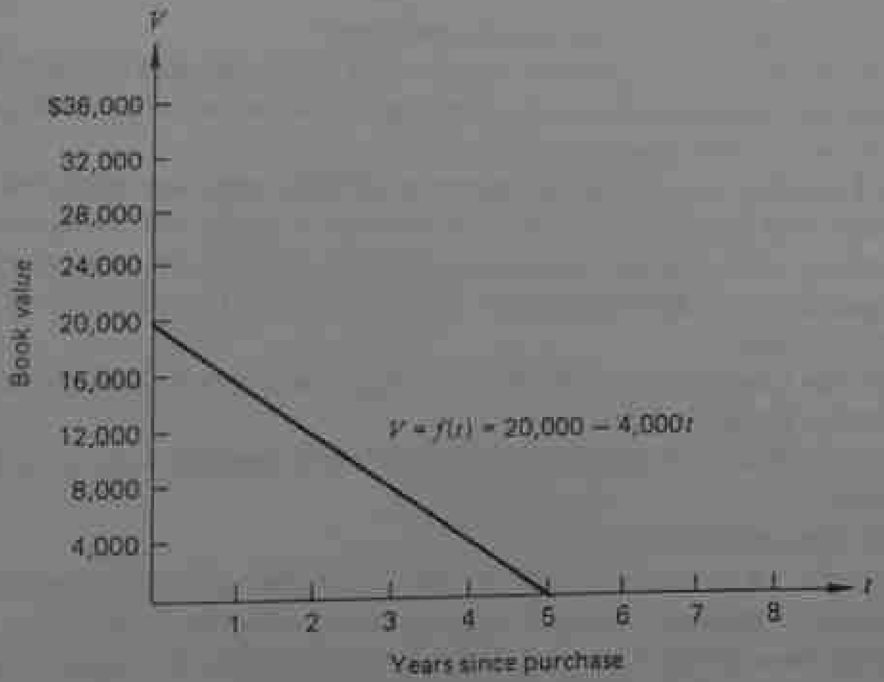

Figure 5.2 Book value function based upon straight-line depreciation.

PRACTICE EXERCISE

Define the restricted domain and range for this function. *Answer:* Domain = $\{t | 0 \le t \le 5\}$; range = $\{V | 0 \le V \le 20{,}000\}$.

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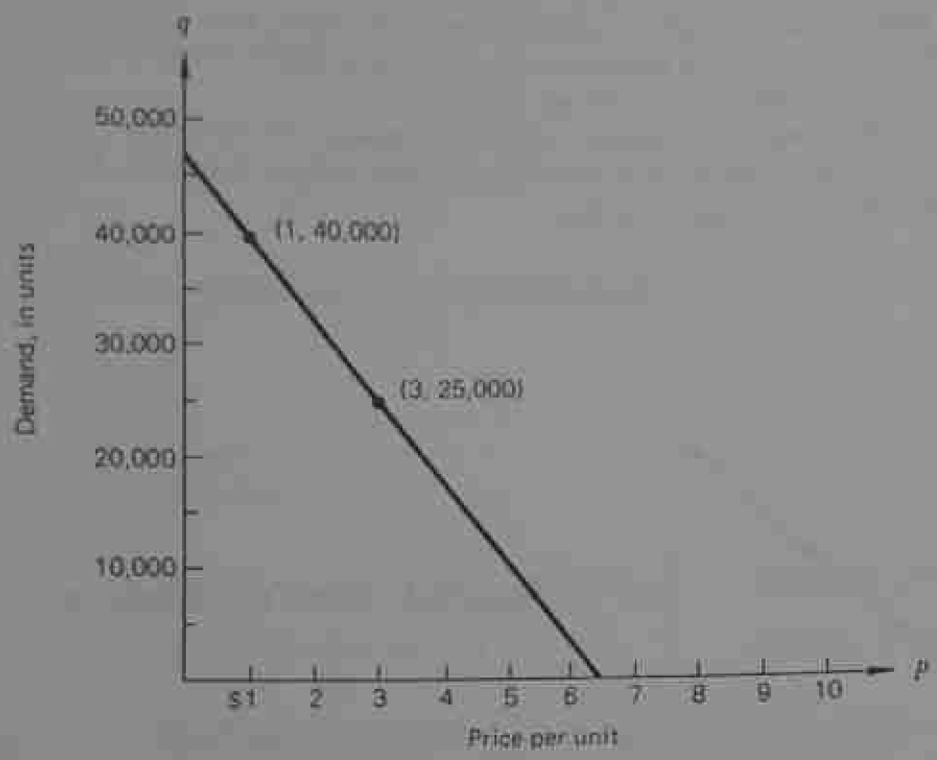
7 **(Linear Demand Functions)** As discussed in Example 13 in Chap. 4, a ***demand function*** is a mathematical relationship expressing the way in which the quantity demanded of an item varies with the price charged for it. The relationship between these two variables—*quantity demanded* and *price per unit*—is usually *inverse*; i.e., a *decrease* in price results in



an *increase* in demand. The purpose of special sales is almost always to stimulate demand. If supermarkets reduced the price of filet mignon to \$0.75 per pound, there would likely be a significant increase in the demand for that item. On the other hand, *increases* in the price of a product usually result in a *decrease* in the demand. The phrase *pricing people out of the market* refers to the customers lost as a result of price increases. If filet mignon were suddenly to triple in price with all other factors such as income levels held constant, many people currently capable of purchasing it would be priced out of the market.

There are exceptions to this behavior, of course. The demand for products or services which are considered *necessities* is likely to fluctuate less with moderate changes in price. Items such as prescription medical drugs, medical services, and certain food items are examples of this class of products.

Although most demand functions are nonlinear, there are situations in which the demand relationship either is, or can be approximated reasonably well by, a linear function. Figure 5.3 illustrates a linear demand function with two sample data points. Although most economics books measure price on the vertical axis and quantity demanded on the horizontal axis, we will reverse the labeling of the axes, as illustrated in Fig. 5.3. The reason for this is that most consumers view the demand relationship as having the form

$$\text{Quantity demanded} = f(\text{price per unit})$$

That is, consumers respond to price. Thus, quantity demanded, the dependent variable, is plotted on the vertical axis.

Verify, using the methods of Chap. 2, that the demand function in Fig. 5.3 has the form

$$q = f(p) = 47{,}500 - 7{,}500p$$

POINT FOR THOUGHT & DISCUSSION Interpret the meaning of the q intercept in this example. Does this seem valid? What is the interpretation of the p intercept? What is the interpretation of the slope in this function?

EXAMPLE 8 **(Linear Supply Functions)** A ***supply function*** relates market price to the quantities that suppliers are willing to produce and sell. The implication of supply functions is that what is brought to the market depends upon the price people are willing to pay. As opposed to the inverse nature of price and quantity demanded, the quantity which suppliers are willing to supply usually varies directly with the market price. *All other factors being equal*, the higher the market price, the more a supplier would like to produce and sell; and the lower the price people are willing to pay, the less the incentive to produce and sell. Assume that you own a lobster boat. All other factors considered equal, how much incentive is there to take your boat and crew out if lobster is wholesaling at \$0.25 per pound? How much incentive is there if it is wholesaling at \$10 per pound?

As with demand functions, supply functions can be approximated sometimes using linear functions. Figure 5.4 illustrates a sample supply function. Note that by labeling the vertical axis q, it is suggested that

$$\text{Quantity supplied} = f(\text{market price})$$

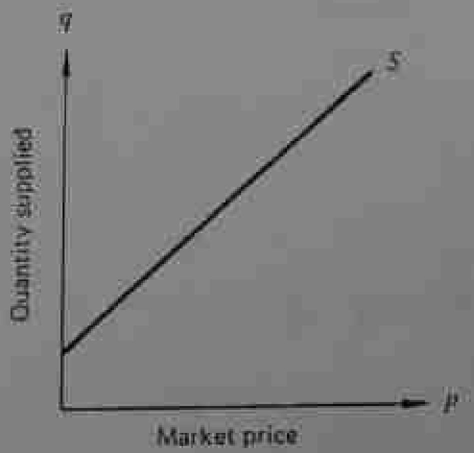


Figure 5.4 Linear supply function.

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POINT FOR THOUGHT & DISCUSSION What does the q intercept in Fig. 5.4 suggest about the relationship between supply and market price? If the supply curve appears as in Fig. 5.5, what does the p intercept suggest about the relationship? Which figure do you believe is the more representative of an actual supply function relationship? Why?

EXAMPLE 9 **(Market Equilibrium: Two Competing Products)** Given supply and demand functions for a product, **market equilibrium** exists if there is a price at which the quantity demanded equals the quantity supplied. This example demonstrates the market equilibrium for two competing products. Suppose that the following demand and supply functions have been estimated for two competing products.

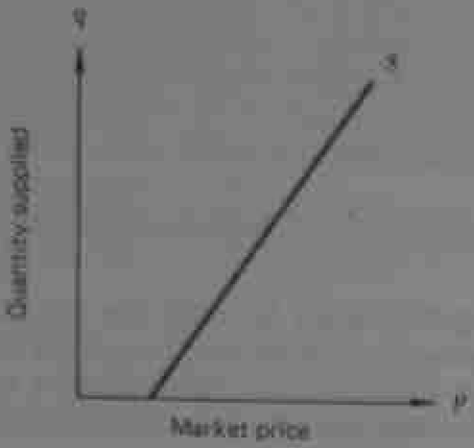

Figure 5.5 Linear supply function.

$$\left.\begin{aligned} q_{d1} &= f_1(p_1, p_2) = 100 - 2p_1 + 3p_2 \\ q_{s1} &= h_1(p_1) = 2p_1 - 4 \end{aligned}\right\} \quad \begin{aligned} &\text{(demand, product 1)} \\ &\text{(supply, product 1)} \end{aligned}$$

$$\left.\begin{aligned} q_{d2} &= f_2(p_1, p_2) = 150 + 4p_1 - p_2 \\ q_{s2} &= h_2(p_2) = 3p_2 - 6 \end{aligned}\right\} \quad \begin{aligned} &\text{(demand, product 2)} \\ &\text{(supply, product 2)} \end{aligned}$$

where q_{d1} = *quantity demanded of product 1*
q_{s1} = *quantity supplied of product 1*
q_{d2} = *quantity demanded of product 2*
q_{s2} = *quantity supplied of product 2*
p_1 = *price of product 1, dollars*
p_2 = *price of product 2, dollars*

Notice that the demand and supply functions are linear. Also note that the quantity demanded of a given product depends not only on the price of the product but also on the price of the competing product. The quantity supplied of a product depends only upon the price of that product.

Market equilibrium would exist in this two-product marketplace if prices existed (and were offered) such that

$$q_{d1} = q_{s1}$$

and

$$q_{d2} = q_{s2}$$

Supply and demand are equal for product 1 when

$$100 - 2p_1 + 3p_2 = 2p_1 - 4$$

or

$$4p_1 - 3p_2 = 104 \tag{5.8}$$

Supply and demand are equal for product 2 when

$$150 + 4p_1 - p_2 = 3p_2 - 6$$

or

$$-4p_1 + 4p_2 = 156 \tag{5.9}$$

If Eqs. (5.8) and (5.9) are solved simultaneously, equilibrium prices are identified as $p_1 = 221$ and $p_2 = 260$. This result suggests that if the products are priced accordingly, the quantities demanded and supplied will be equal for *each* product.

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PRACTICE EXERCISE
Given the prices identified above, calculate q_{d1}, q_{s1}, q_{d2}, and q_{s2}. Are equilibrium conditions satisfied? *Answer:* $q_{d1} = q_{s1} = 438$, $q_{d2} = q_{s2} = 774$; yes.

POINT FOR THOUGHT & DISCUSSION

Explain the logic underlying the assumptions in the demand and supply functions. That is, why is the demand for one product affected by the price of the product as well as the price of the competing product? Explain the logic of the plus sign for the "competing" product in each demand function. What is assumed by the inclusion of only one price variable in the supply functions? Under what circumstances would it be appropriate for both prices to be included in these functions?

EXAMPLE 10 **(Federal Income Taxes: Motivating Scenario)** In 1990, the federal tax rates for a married couple filing jointly were (repeating the table in the Motivating Scenario) as given in Table 5.2. What is desired is a mathematical function which allows a married couple to calculate their tax liability, given their taxable income.

TABLE 5.2 **1990 Federal Tax Rates (Married Filing Jointly)**

Taxable Income		
Over	But not Over	*Tax Rate*
$ 0	$ 32,450	15%
32,450	78,400	28
78,400	162,770	33
162,770		28

SOLUTION

Let x = *taxable income, dollars*

T = *federal income tax liability, dollars*

We want to identify the function

$$T = f(x)$$

First, we must understand the information in Table 5.2. If a couple's taxable income is \$0–\$32,450, they must pay federal income tax equal to 15 percent of the taxable income. If their taxable income is greater than \$32,450 but not greater than \$78,400, they must pay 15 percent on the first \$32,450 and 28 percent on all income over \$32,450. If their taxable income is greater than \$78,400 but not greater than \$162,770, they must pay 15 percent on the first \$32,450, 28 percent on the next \$45,950 (\$78,400 − \$32,450), and 33 percent on all income over \$78,400. Thus, the tax rate applies only to income which falls within the corresponding range.

The tax function will be stated by four component functions, one for each of the taxable income ranges stated in Table 5.2. For example, if taxable income is \$0–\$32,450,

$$T = 0.15x$$

If taxable income is greater than \$32,450 but not more than \$78,400,

$$\begin{aligned} T &= 0.15(32{,}450) + 0.28(x - 32{,}450) \\ &= 4{,}867.5 + 0.28x - 9{,}086 \\ &= 0.28x - 4{,}218.5 \end{aligned}$$

If taxable income is greater than \$78,400 but not more than \$162,770,

$$\begin{aligned} T &= 0.15(32{,}450) + 0.28(45{,}950) + 0.33(x - 78{,}400) \\ &= 4{,}867.5 + 12{,}866 + 0.33x - 25{,}872 \\ &= 0.33x - 8{,}138.5 \end{aligned}$$

If taxable income is greater than \$162,770,

$$\begin{aligned} T &= 0.15(32{,}450) + 0.28(45{,}950) + 0.33(84{,}370) + 0.28(x - 162{,}770) \\ &= 4{,}867.5 + 12{,}866 + 27{,}842.1 + 0.28x - 45{,}575.6 \\ &= 0.28x \end{aligned}$$

The complete tax liability function is

$$T = f(x) = \begin{cases} 0.15x & 0 < x \le 32{,}450 \\ 0.28x - 4{,}218.5 & 32{,}450 < x \le 78{,}400 \\ 0.33x - 8{,}138.5 & 78{,}400 < x \le 162{,}770 \\ 0.28x & 162{,}770 < x \end{cases}$$

Figure 5.6 presents a graph of this tax liability function for persons who filed in the category of "Married, filing jointly" for 1990.

EXAMPLE 11 **(Social Security Taxes)** Figure 5.7 is a graph of Social Security taxes collected in the years 1980–1989. The amount collected annually appeared to be increasing, approximately, at a linear rate. In 1980, the Social Security taxes collected were \$150 billion and in 1989, \$352 billion. Using these two data points, develop the linear function which estimates the Social Security taxes collected as a function of time since 1980

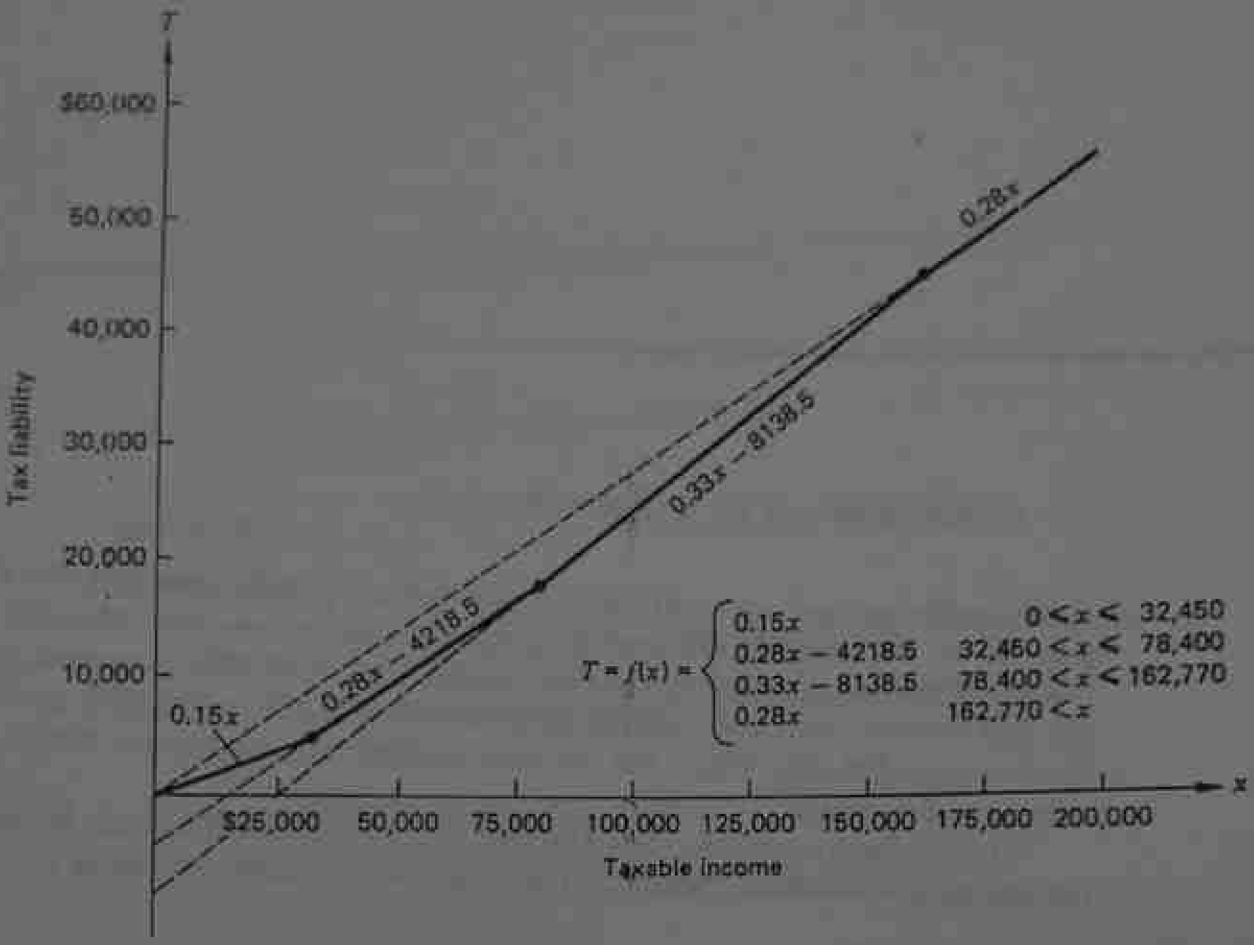


Figure 5.6 1990 tax liability: Married filing jointly.

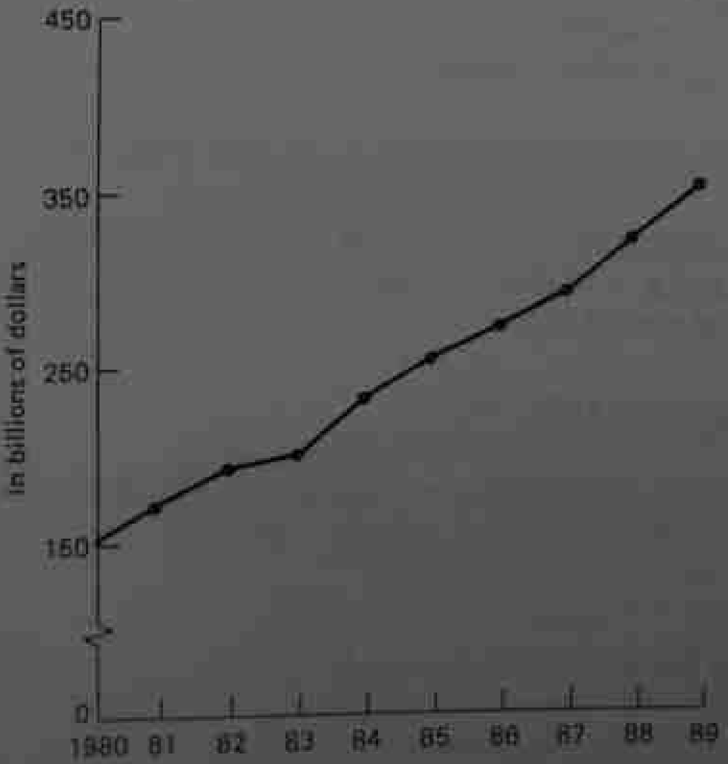


Figure 5.7 Social Security taxes.
(Data: Office of Management & Budget, DRI/McGraw-Hill)

SOLUTION

If we define

S = *Social Security taxes collected, billions of dollars*

t = *time measured in years since 1980*

we want to determine the linear function having the form

$$S = f(t) = a_1 t + a_0$$

The two data points (t, S) are (0, 150) and (9, 352). By observation, the value of a_0 equals 150. Substituting the data point for 1989 into the slope-intercept form gives

$$352 = a_1(9) + 150$$
$$202 = 9a_1$$
$$22.44 = a_1$$

Thus, the linear approximating function is

$$S = f(t) = 22.44t + 150$$

❑

Section 5.2 Follow-up Exercises

1 A piece of machinery is purchased for \$80,000. Accountants have decided to use a straight-line depreciation method with the machine being fully depreciated after 6 years. Letting V equal the book value of the machine and t the age of the machine, determine the function $V = f(t)$. (Assume no salvage value.)

2 **Straight-Line Depreciation with Salvage Value** Many assets have a **resale, or salvage, value** even after they have served the purposes for which they were originally purchased. In such cases, the allocated cost over the life of the asset is the difference between the purchase cost and the salvage value. The cost allocated each time period is the allocated cost divided by the useful life. In Example 6, suppose that it is estimated that the truck (which cost \$20,000) can be resold for \$2,500 at the end of 5 years. The total cost to be allocated over the 5-year period is the purchase cost less the resale value, or \$20,000 − \$2,500 = \$17,500. Using straight-line depreciation, the annual depreciation will be

$$\frac{\text{Purchase cost} - \text{salvage value}}{\text{Useful life (in years)}} = \frac{20{,}000 - 2{,}500}{5} = \frac{17{,}500}{5} = 3{,}500$$

The function expressing the book value V as a function of time t is

$$V = f(t) = 20{,}000 - 3{,}500t \qquad 0 \le t \le 5$$

2 In Exercise 1 assume that the machine will have a salvage value of \$7,500 at the end of 6 years. Determine the function $V = f(t)$ for this situation.

3 A piece of machinery is purchased for \$300,000. Accountants have decided to use a straight-line depreciation method with the machine being fully depreciated after 8 years. Letting V equal the book value of the machine and t the age of the machine, determine the function $V = f(t)$. Assume there is no salvage value.

4 Assume in Exercise 3 that the machine can be resold after 8 years for \$28,000. Determine the function $V = f(t)$.

5 A company purchases cars for use by its executives. The purchase cost this year is \$25,000. The cars are kept 3 years, after which they are expected to have a resale value of \$5,600. If accountants use straight-line depreciation, determine the function which describes the book value V as a function of the age of the car t.

6 A police department believes that arrest rates R are a function of the number of plainclothes officers n assigned. The *arrest rate* is defined as the percentage of cases in which arrests have been made. It is believed that the relationship is linear and that each additional officer assigned to the plainclothes detail results in an increase in the arrest rate of 1.20 percent. If the current plainclothes force consists of 16 officers and the arrest rate is 36 percent:

(*a*) Define the function $R = f(n)$.
(*b*) Interpret the meaning of the R intercept.
(*c*) Determine the restricted domain and range for the function.
(*d*) Sketch the function.

7 Two points on a linear demand function are (\$20, 80,000) and (\$30, 62,500).

(*a*) Determine the demand function $q = f(p)$.
(*b*) Determine what price would result in demand of 50,000 units.
(*c*) Interpret the slope of the function.
(*d*) Define the restricted domain and range for the function.
(*e*) Sketch $f(p)$.

8 Two points (p, q) on a linear demand function are (\$24, 60,000) and (\$32, 44,400).

(*a*) Determine the demand function $q = f(p)$.
(*b*) What price would result in demand of 80,000 units?
(*c*) Interpret the slope of the function.
(*d*) Determine the restricted domain and range.
(*e*) Sketch $f(p)$.

9 Two points on a linear supply function are (\$4.00, 28,000) and (\$6.50, 55,000).

(*a*) Determine the supply function $q = f(p)$.
(*b*) What price would result in suppliers offering 45,000 units?
(*c*) Determine and interpret the p intercept.

10 Two points (p, q) on a linear supply function are (\$3.50, 116,000) and (\$5.00, 180,000).

(*a*) Determine the supply function $q = f(p)$.
(*b*) What price would result in suppliers offering 135,000 units for sale?
(*c*) Interpret the slope of the function.
(*d*) Determine and interpret the p intercept.
(*e*) Sketch $f(p)$.

11 **Alimony/Child Support** Recent surveys indicate that payment of alimony or child support tends to decline with time elapsed after the divorce decree. One survey uses the estimating function

$$p = f(t) = 90 - 12.5t$$

where p equals the percentage of cases in which payments are made and t equals time measured in years after the divorce decree.

(a) Interpret the p intercept.
(b) Interpret the slope.
(c) In what percentage of cases is alimony/child support paid after 5 years?
(d) Sketch $f(t)$.

12 **Sports Injuries** A survey of high school and college football players suggests that the number of career-ending injuries in this sport is increasing. In 1980 the number of such injuries was 925; in 1988 the number was 1,235. If it is assumed that the injuries are increasing at a linear rate:

(a) Determine the function $n = f(t)$, where n equals the estimated number of injuries per year and t equals time measured in years since 1980.
(b) Interpret the meaning of the slope of this function.
(c) When is it expected that the number of such injuries will go over the 1,500 mark?

13 **Marriage Prospects** Data released by the Census Bureau in 1986 indicated the likelihood that never-married women would eventually marry. The data indicated that the older the woman, the less the likelihood of marriage. Specifically, two statistics indicated that women who were 45 and never-married had an 18 percent chance of marriage and women 25 years old had a 78 percent chance of marriage. Assume that a linear fit to these two data points provides a reasonable approximation for the function $p = f(a)$, where p *equals the probability of marriage* and a *equals the age of a never-married woman.*

(a) Determine the linear function $p = f(a)$.
(b) Interpret the slope and p intercept.
(c) Do the values in part b seem reasonable?
(d) If the restricted domain on this function is $20 \leq a \leq 50$, determine $f(20)$, $f(30)$, $f(40)$, and $f(50)$.

14 **Two-Income Families** Figure 5.8 illustrates the results of a survey regarding two-income families. The data reflect the percentage of married couples with wives who work for four different years. The percentage appears to be increasing approximately at a linear rate. Using the data points for 1960 and 1988:

(a) Determine the linear function $P = f(t)$, where P equals the estimated percentage of married couples with wives who work and t equals time measured in years since 1950 *($t = 0$ corresponds to 1950).*

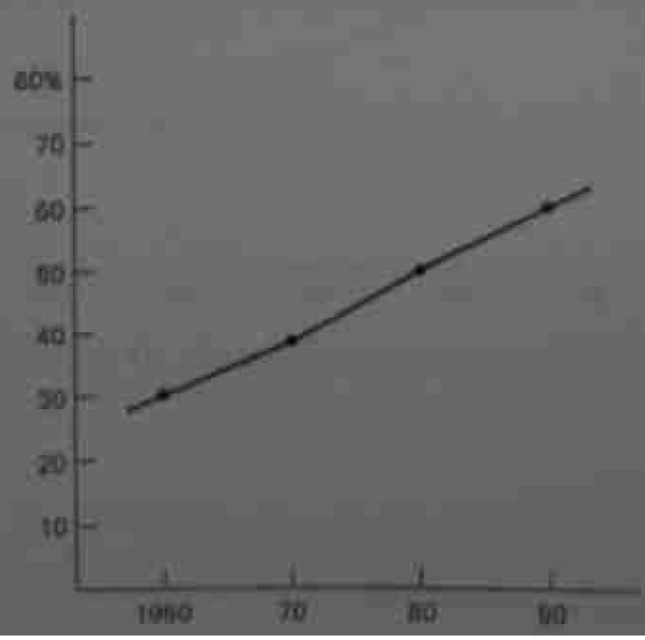


Figure 5.8 Percentage of married couples with wives who work.

(b) Interpret the meaning of the slope and the P intercept.
(c) When is it expected that the percentage will exceed 75 percent?

15 **Education Expenditures** Figure 5.9 illustrates the expenditures per student in U.S. public schools over a three-decade period. The expenditures are stated in "constant dollars" which filter out the effects of inflation. The increase in expenditures per student appears to have occurred approximately at a linear rate. In 1958, the expenditures per student were $1,750; in 1984, the expenditures were $3,812.50. Using these two data points:

(a) Determine the linear approximating function $E = f(t)$, where E equals the estimated expenditure per student in dollars and t equals time measured in years *since* 1955 ($t = 0$ corresponds to 1955).
(b) Interpret the slope and E intercept.
(c) According to this function, what are expenditures per student expected to equal in the year 2000?

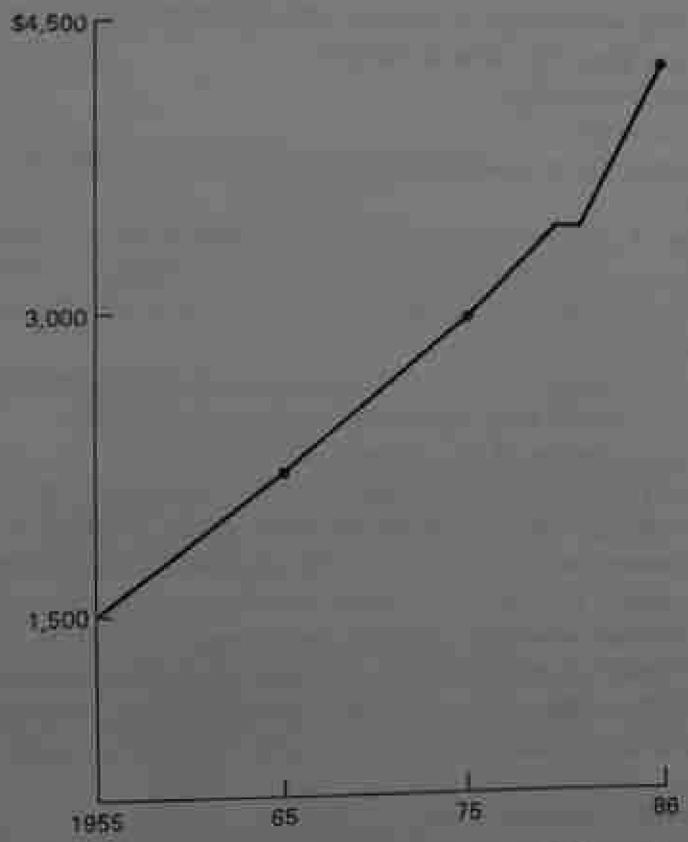


Figure 5.9 Expenditures per student in U.S. public schools (constant dollars). (Source: U.S. Dept. of Education, National Center for Education Statistics)

16 **Walt Disney Company** Figure 5.10 portrays the annual operating profits for Walt Disney Company between 1986 and 1990 (estimated). During this period, annual operating profits appear to have increased approximately in a linear manner. In 1987, annual operating profits were $0.762 billion; in 1989, they were $1.220 billion. Using these two data points:

(a) Determine the linear function $P = f(t)$, where P equals the estimated annual operating profits and t equals time measured in years *since* 1986.
(b) Interpret the slope and P intercept.
(c) Using this function, estimate annual operating profits for Walt Disney Company in the year 2000.

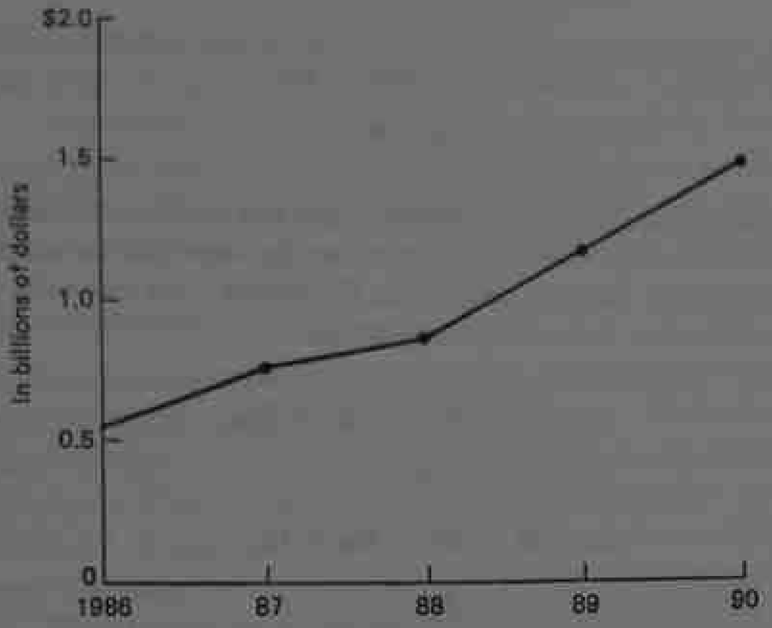


Figure 5.10 Annual operation profits, Walt Disney Company. (Data: Company Reports, Wertheim Schroder & Co.)

17 Economic Downturn A general trend of economic decline within New York City is reflected by Fig. 5.11. This figure indicates the vacancy rate for offices in Manhattan during the period 1985–1990. The increase in the vacancy rate appears to be approximately linear. The vacancy rate in 1986 was 9.4 percent; in 1989, the rate was 13.2 percent. Using these two data points:

(a) Determine the linear function $V = f(t)$, where V equals the estimated vacancy rate (in percent) and t equals time measured in years since 1985.

(b) Interpret the slope and V intercept.

(c) Using the function, estimate the vacancy rate in 1995.

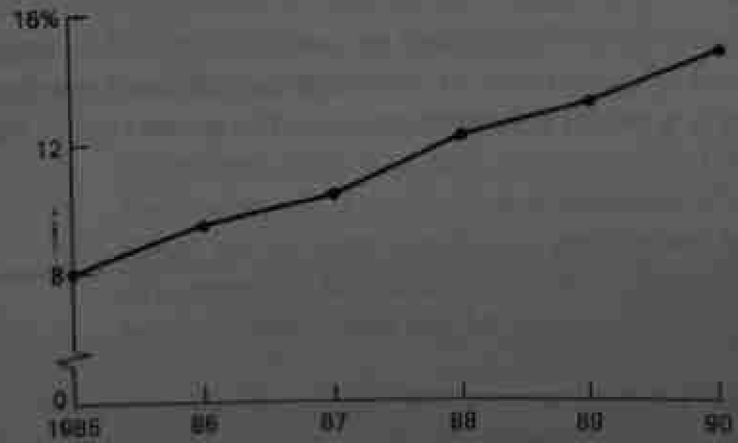


Figure 5.11 Percentage of Manhattan offices vacant. (Business Week, June 18, 1990)

18 Federal Income Taxes Table 5.3 contains the 1990 federal tax rates for a single person. Determine the function $T = f(x)$, where T equals the tax liability (in dollars) for a single person and x equals taxable income (in dollars).

ABLE 5.3

Taxable Income		
Over	But not Over	*Tax Rate*
$ 0	$19,450	15%
19,450	47,050	28
47,050	97,620	33
97,620		28

19 **Market Equilibrium** Given the following demand and supply functions for two competing products,

$$q_{d1} = 82 - 3p_1 + p_2$$
$$q_{s1} = 15p_1 - 5$$
$$q_{d2} = 92 + 2p_1 - 4p_2$$
$$q_{s2} = 32p_2 - 6$$

determine whether there are prices which bring the supply and demand levels into equilibrium for the two products. If so, what are the equilibrium quantities?

*20 **Market Equilibrium: Three Competing Products** The following are the demand and supply functions for three competing products.

$$q_{d1} = 46 - 10p_1 + 2p_2 + 2p_3$$
$$q_{s1} = 12p_1 - 16$$
$$q_{d2} = 30 + 2p_1 - 6p_2 + 4p_3$$
$$q_{s2} = 6p_2 - 22$$
$$q_{d3} = 38 + 2p_1 + 4p_2 - 8p_3$$
$$q_{s3} = 6p_3 - 10$$

Determine whether there are prices which would bring the supply and demand levels into equilibrium for each of the three products. If so, what are the equilibrium demand and supply quantities?

5.3 BREAK-EVEN MODELS

In this section we will discuss ***break-even models,*** a set of planning tools which can be, and has been, very useful in managing organizations. One significant indication of the performance of a company is reflected by the so-called bottom line of the income statement for the firm; that is, how much profit is earned! Break-even analysis focuses upon the profitability of a firm. Of specific concern in break-even analysis is identifying the level of operation or level of output that would result in a **zero profit.** This level of operations or output is called the ***break-even point.*** The break-even point is a useful reference point in the sense that it represents the level of operation at which **total revenue equals total cost.** Any changes from this level of operation will result in either a profit or a loss.

Break-even analysis is valuable particularly as a planning tool when firms are contemplating expansions such as offering new products or services. Similarly, it is useful in evaluating the pros and cons of beginning a new business venture. In each instance the analysis allows for a projection of profitability.

Assumptions

In this discussion we will focus upon situations in which both the total cost function and the total revenue function are linear. The use of a linear total cost function implies that variable costs per unit either are constant or can be assumed to be constant. The linear cost function assumes that total variable costs depend upon the level of operation or output. It is also assumed that the fixed-cost portion of the cost function is constant over the level of operation or output being considered.

The linear total revenue function assumes that the selling price per unit is constant. Where the selling price is not constant, average price is sometimes chosen for purposes of conducting the analysis.

Another assumption is that price per unit is greater than variable cost per unit. *Think about that for a moment.* If price per unit is less than variable cost per unit, a firm will lose money on every unit produced and sold. A break-even condition could never exist.

Break-Even Analysis

In break-even analysis the primary objective is to determine the break-even point. The break-even point may be expressed in terms of (1) *volume of output* (or level of activity), (2) *total dollar sales,* or possibly (3) *percentage of production capacity.* For example, it might be stated that a firm will break even at 100,000 units of output, when total sales equal $2.5 million or when the firm is operating at 60 percent of its plant capacity. We will focus primarily on the first of these three ways.

The methods of performing break-even analysis are rather straightforward, and there are alternative ways of determining the break-even point. The usual approach is as follows:

1. *Formulate total cost as a function of x, the level of output.*
2. *Formulate total revenue as a function of x.*
3. *Since break-even conditions exist when total revenue equals total cost, set $C(x)$ equal to $R(x)$ and solve for x. The resulting value of x is the break-even level of output and might be denoted by x_{BE}.*

An alternative to step 3 is to construct the profit function $P(x) = R(x) - C(x)$, set $P(x)$ equal to zero, and solve for x_{BE}.

The following example illustrates both approaches.

LE 12 A group of engineers is interested in forming a company to produce smoke detectors. They have developed a design and estimate that variable costs per unit, including materials, labor, and marketing costs, are $22.50. Fixed costs associated with the formation, operation, and management of the company and the purchase of equipment and machinery total $250,000.

They estimate that the selling price will be \$30 per detector.
(*a*) Determine the number of smoke detectors which must be sold in order for the firm to break even on the venture.
(*b*) Preliminary marketing data indicate that the firm can expect to sell approximately 30,000 smoke detectors over the life of the project if the detectors are sold for \$30 per unit. Determine expected profits at this level of output.

SOLUTION

(*a*) If x equals the number of smoke detectors produced and sold, the total revenue function is represented by the equation

$$R(x) = 30x$$

The total cost function is represented by the equation

$$C(x) = 22.50x + 250{,}000$$

The break-even condition occurs when total revenue equals total cost, or when

$$\boxed{R(x) = C(x)} \tag{5.10}$$

For this problem the break-even point is computed as

$$30x = 22.50x + 250{,}000$$

or

$$7.50x = 250{,}000$$

and

$$x_{BE} = 33{,}333.33 \text{ units}$$

The alternative approach is first to write the profit function and set it equal to 0, as follows:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 30x - (22.50x + 250{,}000) \\ &= 7.50x - 250{,}000 \end{aligned}$$

Setting the profit function P equal to 0, we have

$$7.50x - 250{,}000 = 0$$

$$7.50x = 250{,}000$$

or

$$x_{BE} = 33{,}333.33 \text{ units}$$

This is the same result, and our conclusion is that *given the assumed cost and price parameters (values),* the firm must sell 33,333.33 units in order to break even.

PRACTICE EXERCISE

Verify that total revenue and total costs both equal \$1,000,000 (taking rounding into account) at the break-even point.

(*b*) With sales projected at 30,000 smoke detectors,

$$P(30{,}000) = 7.5(30{,}000) - 250{,}000$$
$$= 225{,}000 - 250{,}000 = \underline{-25{,}000}$$

This suggests that if all estimates—price, cost, and demand—hold true, the firm can expect to lose \$25,000 on the venture.

EXAMPLE 13

(Graphical Approach) The essence of break-even analysis is illustrated quite effectively by graphical analysis. Figure 5.12*a* illustrates the total revenue function, Fig. 5.12*b*, the total cost function, and Fig. 5.12*c*, a composite graph showing both functions, for Example 12. Note in Fig. 5.12*b* that the fixed-cost component is distinguished from the variable-cost component. At *any* level of output x, the vertical distance within the darker shaded area indicates the fixed cost of \$250,000. To this is added the total variable cost, which is represented by the vertical distance at x within the lighter area. The sum of these two vertical distances represents the total cost $C(x)$.

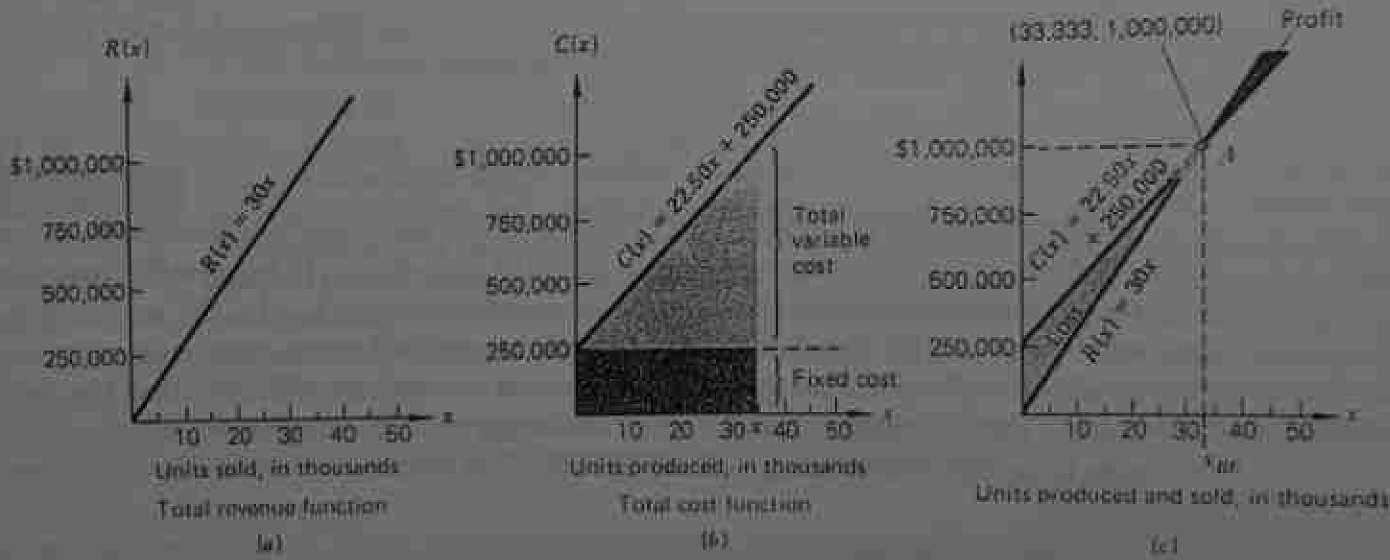
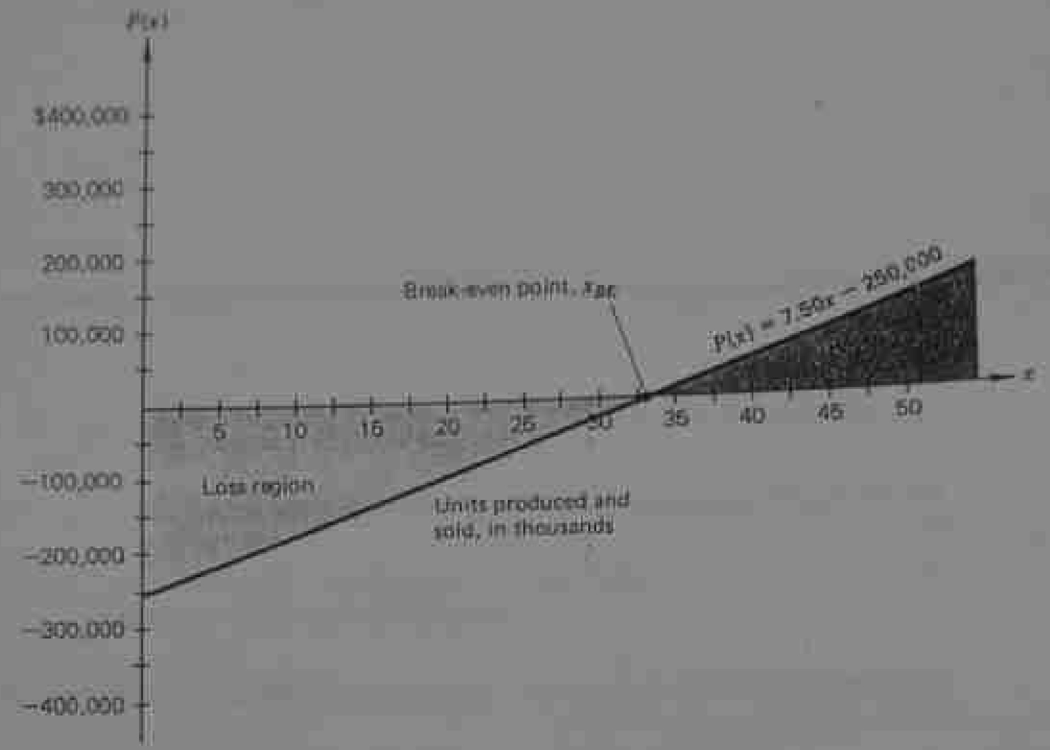


Figure 5.12

In Fig. 5.12*c* the two functions are graphed on the same set of axes. The point where the two functions intersect represents the one level of output where total revenue and total cost are equal. This is the break-even point. For all points to the left of the break-even point the cost function C has a value greater than the revenue function R. In this region the vertical distance separating the two functions represents the loss which would occur at a given level of output. To the right of $x = 33.333$, $R(x)$ is higher than $C(x)$, or $R(x) > C(x)$. For levels of output greater than $x = 33.333$ the vertical distance separating $R(x)$ and $C(x)$ represents the profit at a given level of output.

Figure 5.13 illustrates the profit function P for this example. The break-even point is identified by the x coordinate of the x intercept. Note that to the left of the break-even point the profit function is below the x axis, indicating a negative profit, or loss. To the right, $P(x)$ is above the x axis, indicating a positive profit.

Figure 5.13
Profit function



POINT FOR THOUGHT & DISCUSSION	Discuss any changes in Fig. 5.13 and the break-even point if (*a*) the price per unit increases (decreases), (*b*) the fixed cost increases (decreases), and (*c*) the variable cost per unit increases (decreases).

An alternative way of viewing break-even analysis is in terms of ***profit contribution.*** As long as the price per unit p exceeds the variable cost per unit v, the sale of each unit results in a contribution to profit. The difference between p and v is called the ***profit margin.*** Or stated in equation form,

$$\text{Profit margin} = p - v \qquad p > v \tag{5.11}$$

The profit margin generated from the sale of units must first be allocated to recover any fixed costs which exist. At lower levels of output, the ***total profit contribution*** (profit margin for all units sold) is typically less than fixed costs, meaning that total profit is negative (see Fig. 5.13). Only when *total* profit contribution exceeds fixed cost will a positive profit exist. Because of this orientation — that the profit margin per unit contributes first to recovering fixed costs, after which it contributes to profit — profit margin is often called the ***contribution to fixed cost and profit.***

With this perspective in mind, the computation of the break-even point can be thought of as determining the number of units to produce and sell in order to recover the fixed costs. The calculation of the break-even point is thus

$$\text{Break-even level of output} = \frac{\text{fixed cost}}{\text{contribution to fixed cost and profit}}$$

or

$$\boxed{x_{BE} = \frac{FC}{p - v}} \qquad (5.12)$$

In fact, if you solve break-even problems by setting $R(x) = C(x)$ or by setting profit $P(x)$ equal to zero, the eventual calculation performed is that of Eq. (5.12). Applying Eq. (5.12) to Example 12 gives

$$\begin{aligned} x_{BE} &= \frac{250{,}000}{30.000 - 22.50} \\ &= \frac{250{,}000}{7.5} \\ &= 33{,}333.33 \end{aligned}$$

If you look back at Example 12, the final computation reduced to the one above, regardless of the approach taken.

EXAMPLE 14 **(Convention Planning)** A professional organization is planning its annual convention to be held in San Francisco. Arrangements are being made with a large hotel in which the convention will be held. Registrants for the 3-day convention will be charged a flat fee of $500 per person, which includes registration fee, room, all meals, and tips. The hotel charges the organization $20,000 for the use of the facilities such as meeting rooms, ballroom, and recreational facilities. In addition, the hotel charges $295 per person for room, meals, and tips. The professional organization appropriates $125 of the $500 fee as annual dues to be deposited in the treasury of the national office. Determine the number of registrants necessary for the organization to recover the fixed cost of $20,000.

SOLUTION

The contribution to fixed cost and profit is the registration fee (price) per person less the cost per person charged by the hotel less the national organization's share per registrant, or

$$\begin{aligned} \text{Contribution per registrant} &= \text{registration fee} - \text{hotel charge per person} - \text{annual dues} \\ &= 500 - 295 - 125 = \$80 \end{aligned}$$

Therefore, according to Eq. (5.12), the number of registrants required to recover the fixed cost is

$$x_{BE} = \frac{20{,}000}{80} = 250 \text{ persons}$$

EXAMPLE 15 **(The Movie "Dick Tracy")** The movie "Dick Tracy," starring Warren Beatty and Madonna, was released in the summer of 1990. It was estimated that the movie cost the Walt Disney Company $45 million to produce and market. It was estimated that the movie would have to gross $100 million at the box office to "break even." *What percentage of the box office gross was Disney expecting to earn on this movie?*

SOLUTION

This example requires a slightly different type of analysis related to the break-even concept. If *x equals the percentage of the box office gross*, the information we are given is that Disney would break even if the box office gross equalled $100 million. The costs they need to recover are the $45 million production and marketing costs. Therefore, to break even

$$(\text{Box office gross})(\text{Disney \% of gross}) = 45 \text{ million}$$

or,

$$100x = 45$$

$$x = 45/100 = 0.45$$

Thus, Disney's share of the box office gross must equal 45%.

PLE 16 **(In-House Computer vs. Service Bureau Decision)** A large medical group practice has 30 full-time physicians. Currently, all billing of patients is done manually by clerks. Because of the heavy volume of billing the business manager believes it is time to convert from manual to computerized patient billing. Two options are being considered: (1) the group practice can lease its own computer and software and do the billing itself or (2) the group can contract with a computer service bureau which will do the patient billing.

The costs of each alternative are a function of the number of patient bills. The lowest bid submitted by a service bureau would result in an annual flat fee of $3,000 plus $0.95 per bill processed. With the help of a computer consultant, the business manager has estimated that the group can lease a small business computer system and the required software at a cost of $15,000 per year. Variable costs of doing the billing in this manner are estimated at $0.65 per bill.

If *x equals the number of patient bills per year*, the annual billing cost using a service bureau is represented by the function

$$S(x) = 3{,}000 + 0.95x$$

The annual cost of leasing a computer system and doing the billing in-house is expressed by the function

$$L(x) = 15{,}000 + 0.65x$$

These two alternatives are equally costly when

$$S(x) = L(x)$$

or

$$3{,}000 + 0.95x = 15{,}000 + 0.65x$$

$$0.30x = 12{,}000$$

$$x = 40{,}000$$

Thus, if the expected number of patient bills per year exceeds 40,000, the lease option is the

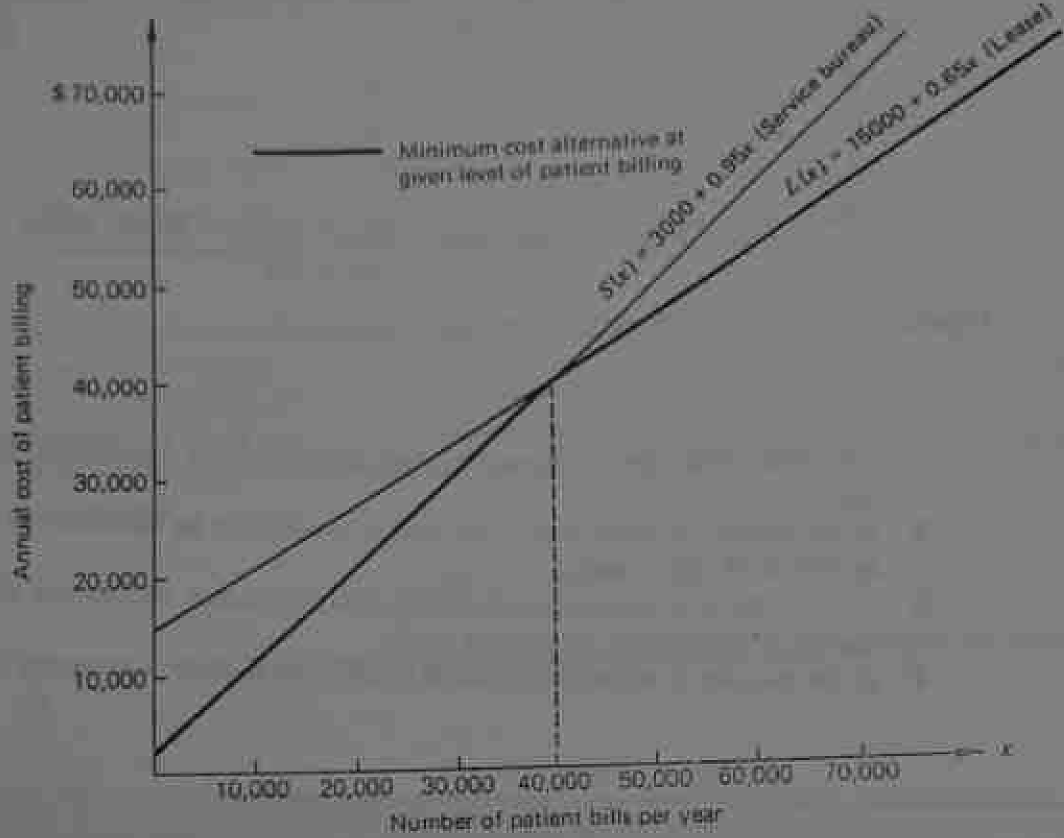


Figure 5.14 Patient-billing cost functions: two options.

less costly. If the number of patient bills is expected to be less than 40,000, the service bureau option is the less costly. Figure 5.14 illustrates the two cost functions.

❑

POINT FOR THOUGHT & DISCUSSION Suppose the patient-billing volume is expected to be 35,000 per year. What reasons favoring the *lease* option could you present to the business manager? Discuss potential advantages and disadvantages which are not quantifiable for the *lease* and *service bureau* options.

EXAMPLE 17 **(Patient-Billing Revisited: Three Alternatives)** Suppose in the previous example that the business manager is not convinced that computer processing is the most cost-effective means of handling patient billing. She estimates that processing bills manually costs the group practice \$1.25 per bill, or

$$M(x) = 1.25x$$

If this method is considered as a third option, let's examine the implications. The three cost functions are graphed together in Fig. 5.15. If you study this figure carefully, you should reach the following conclusions:

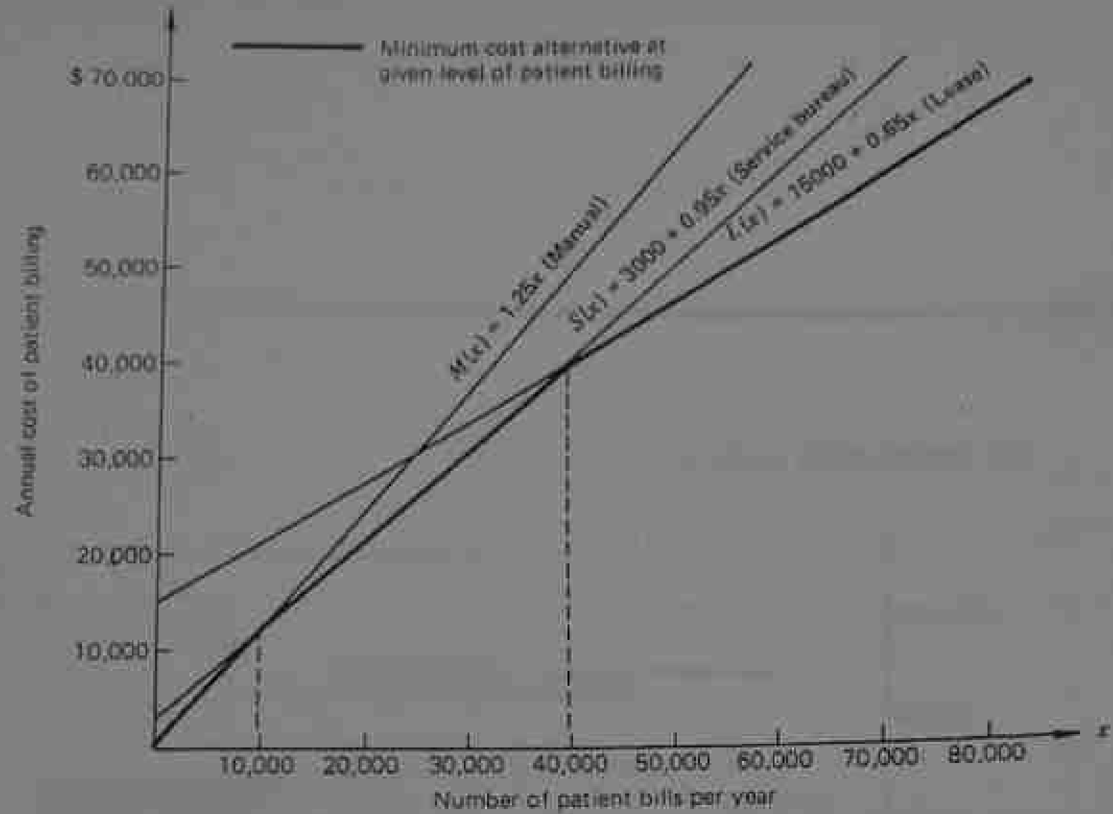


Figure 5.15 Patient-billing cost functions: three options.

1 *The least costly option at any level of patient bills is highlighted by the heavy line segments.*
2 *If the number of patient bills per year is expected to be less than 10,000, the manual system is the least costly.*
3 *If the number of patient bills is expected to be between 10,000 and 40,000, the service bureau arrangement is the least costly.*
4 *If the number is expected to exceed 40,000, the lease arrangement is the least costly.*

☐

NOTE When performing break-even analysis for more than two alternatives (as in this example), it is strongly advised that you first sketch the relevant functions. Sometimes, the interaction which occurs between such functions is not always what might be expected. A sketch will quickly show the interaction.

E 18 **(Multiproduct Analysis)** Our discussion in this section has been limited to single-product/service situations. For multiproduct situations, break-even analysis can be performed when a ***product mix*** is known. The product mix expresses the ratio of output levels for the different products. For example, a firm having three products might produce 3 units of product A and 2 units of product B for each unit of product C. In this situation we might define *1 **unit of product mix*** as consisting of 3 units of product A, 2 units of B, and 1 unit of

5.4

	Product		
	A	B	C
Price/unit	$40	$30	$55
Variable cost/unit	30	21	43
Profit margin	$10	$ 9	$12

C. If a product mix unit can be defined, we can conduct break-even analysis using this as the measure of output.

Suppose that these three products have the price and cost attributes shown in Table 5.4. Combined fixed cost for the three products is $240,000. Since *1 unit of the product mix* consists of 3 units of *A*, 2 units of *B*, and 1 unit of *C*, the *profit contribution per unit of product mix equals*

$$3(\$10) + 2(\$9) + 1(\$12) = \$60$$

If we let *x equal the number of units of product mix*, the profit function for the three products is

$$P(x) = 60x - 240{,}000$$

The break-even point occurs when $P(x) = 0$, or

$$60x - 240{,}000 = 0$$
$$60x = 240{,}000$$
$$x = 4{,}000$$

The firm will break even when it produces 4,000 product mix units, or 12,000 units of *A*, 8,000 units of *B*, and 4,000 units of *C*.

□

The analysis presented in Example 18 presumes that a product mix is known. If the product mix is not known exactly but can be approximated, this analysis can still be of value as a planning tool.

Section 5.3 Follow-up Exercises

1 A firm sells a product for $45 per unit. Variable costs per unit are $33 and fixed costs equal $450,000. How many units must be sold in order to break even?

2 An enterprising college student has decided to purchase a local car wash business. The purchase cost is $150,000. Car washes will be priced at $5.50, and variable cost per car (soap, water, labor, etc.) is expected to equal $1.50. How many cars must be washed in order to recover the $150,000 purchase price?

3 A charitable organization is planning a raffle to raise $10,000. Five hundred chances will be sold on a new car. The car will cost the organization $15,000. How much should each ticket cost if the organization wishes to net a profit of $10,000?

4 A publisher has a fixed cost of \$250,000 associated with the production of a college mathematics book. The contribution to profit and fixed cost from the sale of each book is \$6.25.
   (a) Determine the number of books which must be sold in order to break even.
   (b) What is the expected profit if 50,000 books are sold?

5 A local university football team has added a national power to next year's schedule. The other team has agreed to play the game for a guaranteed fee of \$100,000 plus 25 percent of the gate receipts. Assume the ticket price is \$12.
   (a) Determine the number of tickets which must be sold to recover the \$100,000 guarantee.
   (b) If college officials hope to net a profit of \$240,000 from the game, how many tickets must be sold?
   (c) If a sellout of 50,000 fans is assured, what ticket price would allow the university to earn the desired profit of \$240,000?
   (d) Again assuming a sellout, what would total profit equal if the \$12 price is charged?

6 **Make or Buy Decision** Assume that a manufacturer can purchase a needed component from a supplier at a cost of \$9.50 per unit, or it can invest \$60,000 in equipment and produce the item at a cost of \$7.00 per unit.
   (a) Determine the quantity for which total costs are equal for the *make* and *buy* alternatives.
   (b) What is the minimum cost alternative if 15,000 units are required? What is the minimum cost?
   (c) If the number of units required of the component is close to the break-even quantity, what factors might influence the final decision to make or buy?

7 A local civic arena is negotiating a contract with a touring ice-skating show, Icey Blades. Icey Blades charges a flat fee of \$60,000 per night plus 40 percent of the gate receipts. The civic arena plans to charge one price for all seats, \$12.50 per ticket.
   (a) Determine the number of tickets which must be sold each night in order to break even.
   (b) If the civic arena has a goal of clearing \$15,000 each night, how many tickets must be sold?
   (c) What would nightly profit equal if average attendance is 7,500 per night?

8 In the previous exercise, assume that past experience with this show indicates that average attendance should equal 7,500 persons.
   (a) What ticket price would allow the civic arena to break even?
   (b) What ticket price would allow them to earn a profit of \$15,000?

9 **Equipment Selection** A firm has two equipment alternatives it can choose from in producing a new product. One automated piece of equipment costs \$200,000 and produces items at a cost of \$4 per unit. Another semiautomated piece of equipment costs \$125,000 and produces items at a cost of \$5.25 per unit.
   (a) What volume of output makes the two pieces of equipment equally costly?
   (b) If 80,000 units are to be produced, which piece of equipment is less costly? What is the minimum cost?

10 **Robotics** A manufacturer is interested in introducing the robotics technology into one of its production processes. The process is one which would provide a "hostile environment" for humans. To be more specific, the process involves exposure to extremely high temperatures as well as to potentially toxic fumes. Two robots which appear to have the capabilities for executing the functions of the production process have been identified. There appear to be no significant differences in the speeds at which the two models work. One robot costs \$180,000 and has estimated maintenance

costs of $100 per hour of operation. The second type of robot costs $250,000 with maintenance costs estimated at $80 per hour of operation.

(a) At what level of operation (total production hours) are the two robots equally costly? What is the associated cost?

(b) Define the levels of operation for which each robot would be the less costly.

11 **Computer Software Development** A firm has a computer which it uses for a variety of purposes. One of the major costs associated with the computer is software development (writing computer programs). The vice president for information systems wants to evaluate whether it is less costly to have his own programming staff or to have programs developed by a software development firm. The costs of both options are a function of the number of lines of code (program statements). The vice president estimates that in-house development costs $1.50 per line of code. In addition, annual overhead costs for supporting the programmers equal $30,000. Software developed outside the firm costs, on average, $2.25 per line of code.

(a) How many lines of code per year make costs of the two options equal?

(b) If programming needs are estimated at 30,000 lines per year, what are the costs of the two options?

(c) In part *b* what would the in-house cost per line of code have to equal for the two options to be equally costly?

12 **Sensitivity Analysis** Because the parameters (constants) used in mathematical models are frequently estimates, actual results may differ from those projected by the mathematical analysis. To account for some of the uncertainties which may exist in a problem, analysts often conduct *sensitivity analysis*. The objective is to assess how much a solution might change if there are changes in model parameters.

Assume in the previous exercise that software development costs by outside firms might actually fluctuate by ±20 percent from the $2.25 per-line estimate.

(a) Recompute the break-even point if costs are 20 percent higher or lower and compare your result with the original answer.

(b) Along with the uncertainty in part *a*, in-house variable costs might increase by as much as 30 percent because of a new union contract. Determine the combined effects of these uncertainties.

13 **Video Games** A leading manufacturer of video games is about to introduce four new games. The accompanying table summarizes price and cost data. Combined fixed costs equal $500,000. A marketing research study predicts that for each unit sold of Black Hole, 1.5 units of Haley's Comet, 3 units of Astervoids, and 4 units of PacPerson will be sold.

(a) How many product mix units must be sold to break even?

(b) How does this translate into sales of individual games?

	Game			
	PacPerson	Astervoids	Haley's Comet	Black Hole
Selling price	$50	$45	$30	$20
Variable cost/unit	20	15	10	10

14 A company produces three products which sell in a ratio of 4 units of product 2 and 2 units of product 3 for each unit sold of product 1. The following table summarizes price and cost data for the three products. If fixed costs are estimated at $2.8 million, determine the number of units of each product needed to break even

	Product		
	1	2	3
Selling price	$40	$32	$55
Variable cost/unit	20	24	46

*15 A company is considering the purchase of a piece of equipment to be used to manufacture a new product. Four machines are being considered. The following table summarizes the purchase cost of each machine and the associated variable cost of production if the machine is used to produce the new product. Determine the ranges of output over which each machine would be the least costly alternative. Sketch the four cost functions.

	Purchase Cost	Variable Cost/Unit
Machine 1	$ 80,000	$10.00
Machine 2	120,000	9.00
Machine 3	200,000	7.50
Machine 4	300,000	5.50

❑ KEY TERMS AND CONCEPTS

❑ IMPORTANT FORMULAS

$$y = f(x) = a_1x + a_0 \qquad (5.1)$$

$$y = f(x_1, x_2) = a_1x_1 + a_2x_2 + a_0 \qquad (5.2)$$

$$y = f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_0 \qquad (5.3)$$

$$P(x) = R(x) - C(x) \qquad (5.7)$$

$$\text{Profit margin} = p - v \qquad p > v \qquad (5.11)$$

Break-even conditions:

$$R(x) = C(x) \qquad (5.10)$$

$$x_{BE} = \frac{FC}{p - v} \qquad (5.12)$$